Fractals Rule!

Some Background Information Helpful to People Wanting to Understand Fractal Geometry

Background Concepts

- Iterative Sequences
- Seed Values
- Convergence and Divergence
- Complex Numbers
- Complex Plane
- Multiplying Complex Numbers
- Mandelbrot Set and Julia Sets

Iterative Sequences

- **Feed a number to a function and get the output.**
- **Apply the same function to output as new input.**
- **Continue…**

$$\{x, f(x), f(f(x)), f(f(f(x))), f(f(f(f(x)))), \ldots\}$$

The *orbit* of x – Converges? Diverges? Cycles?

Seed Values: An Example

$A_{n+1} = A_n + A_{n-1}$

Seed Values: $A_0 = 0$ $A_1 = 1$

$A_2 = 0 + 1 = 1$

$A_3 = 1 + 1 = 2$

$A_4 = 2 + 1 = 3$

$A_5 = 3 + 2 = 5$

Sequence:
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89…

Sequences in General

Classic examples:

Figurate Numbers

- Triangular Numbers: 1, 3, 6, 10… $n(n+1)/2$
- Square Numbers: 1, 4, 9, 25, 36… n^2
- Cuboctahedral #s: 1, 12, 42, 92… 1 ; $10n^2+2$

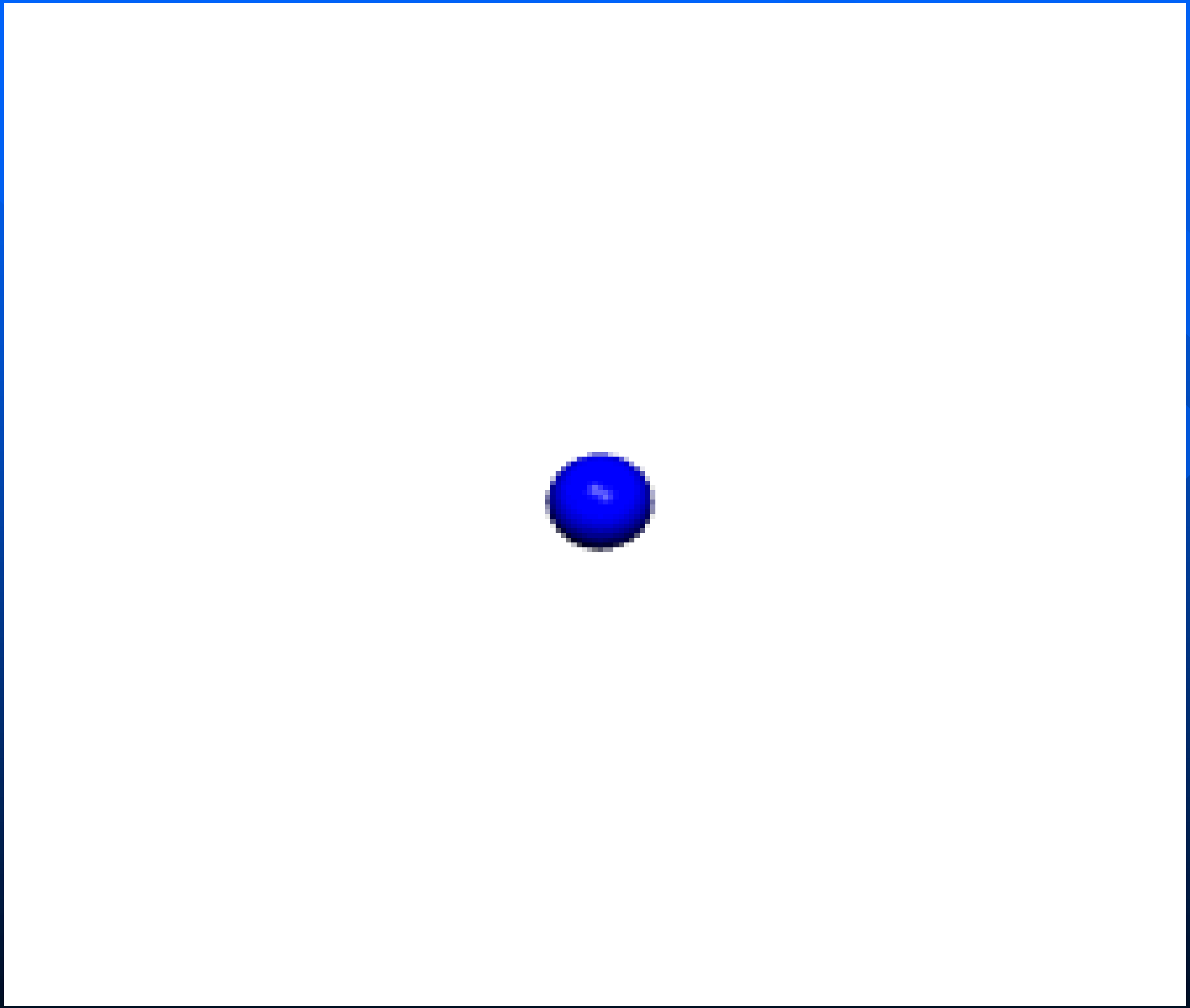
Primes P(n): 2,3,5,7,11,13,17,19,23,29…

(sometimes no rule is known)

See:

Sloane's On-Line Dictionary of Integer Sequences

Cuboctahedral Numbers



Convergence / Divergence

Sequences defined by simple rules:

- Additive (e.g. $A_{n+1} = A_n + A_{n-1}$)
- Geometric (e.g. $z_n = r\, z_{n-1}$: 2, 4, 8, 16,…)
- Other ($z_n = z_{n-1}^2 + c$)

Sequences defined by partial sums: running totals of elements

Divergent sequences: {1, 1+1/2, 1+1/2+1/3, 1+1/2+1/3+1/4, …}

Convergent sequences: {1, 1+1/4, 1+1/4+1/9, 1+1/4+1/9+1/16,…}

Cycling sequences: {decimal digits of 1/7}

Chaotic or aperiodic sequences: {decimal digits of PI}

Side Topic: Converging Ratios

Sequence of Fibonacci #s:
0,1,1,2,3,5,8,13,21,34,55…

Ratios:
1/1, 2/1, 3/2, 5/3, 8/5, 13/8…

Fib_{n+1}/Fib_n goes to ♐ as n goes to infinity

♐ = (1 + ⌧🗄✆📪🗎

Fibonacci sequence is both additive *and* geometric

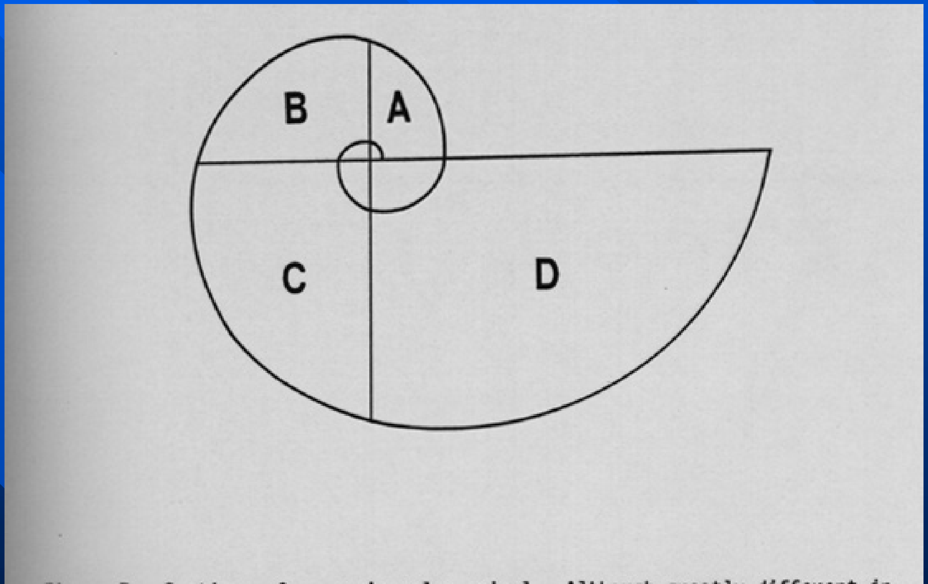


Figure 7. Sections of an equiangular spiral. Although greatly different in size, sections A, B, C and D are identical in shape. (After Davis and Hersh, 1986.)

Self Similarity around a single point

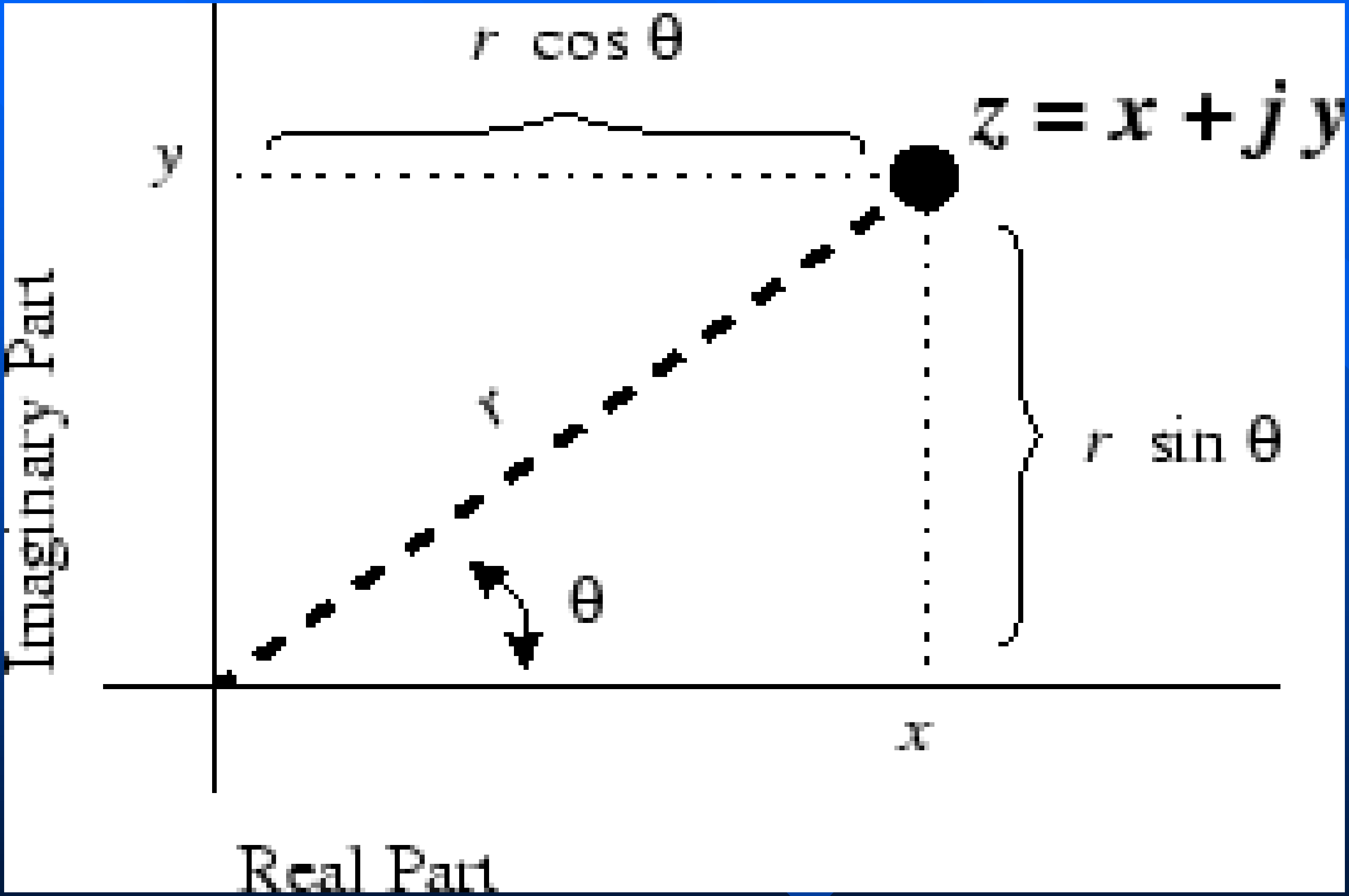
Complex Numbers

Caspar Wessel (1745-1818)
Jean –Robert Argand (1768-1822)
Abraham De Moivre (1667-1754)
Carl Friedrich Gauss (1777 – 1855)

Solutions to polynomials sometimes involved complex quantities even when all the solutions were real. Troubling to many.

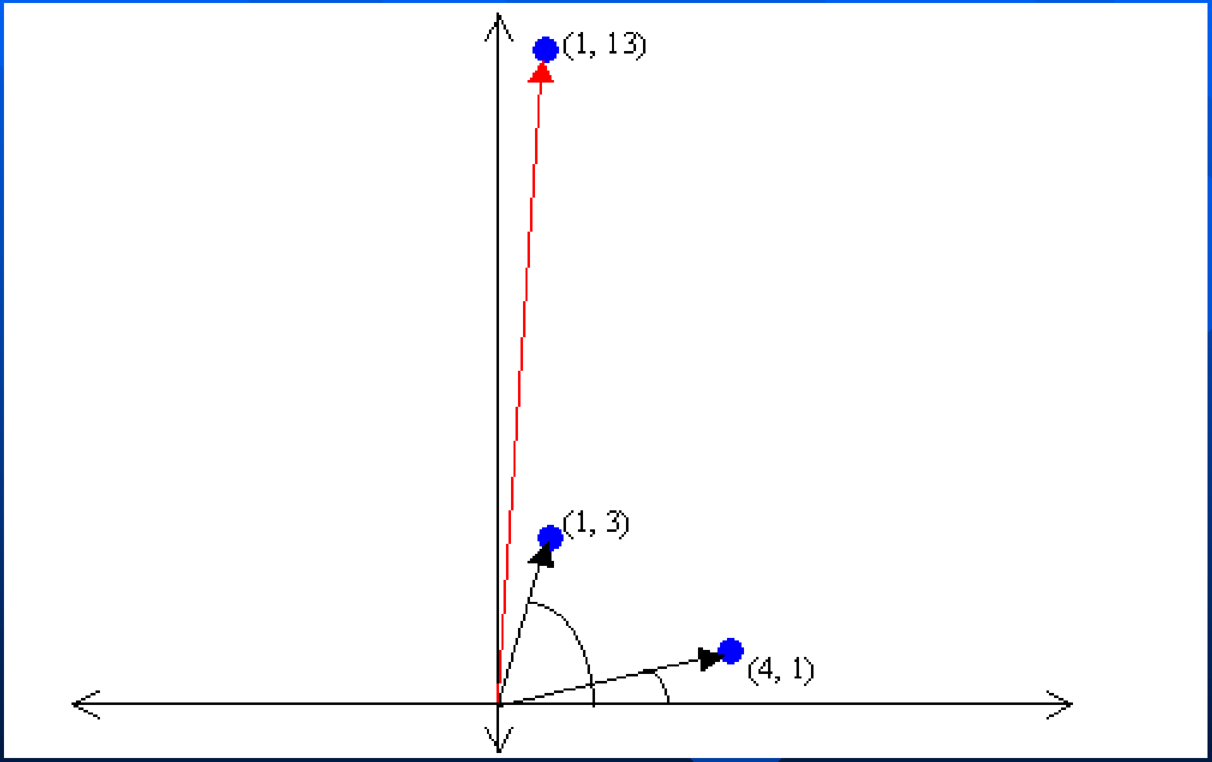
Generalization of polynomials required expansion of real numbers to this larger set allowing –1 to have even numbers of roots.

Argand Plane



$Z_1 \times Z_2$

Multiply Absolute Values, Add angles



More on Multiplying

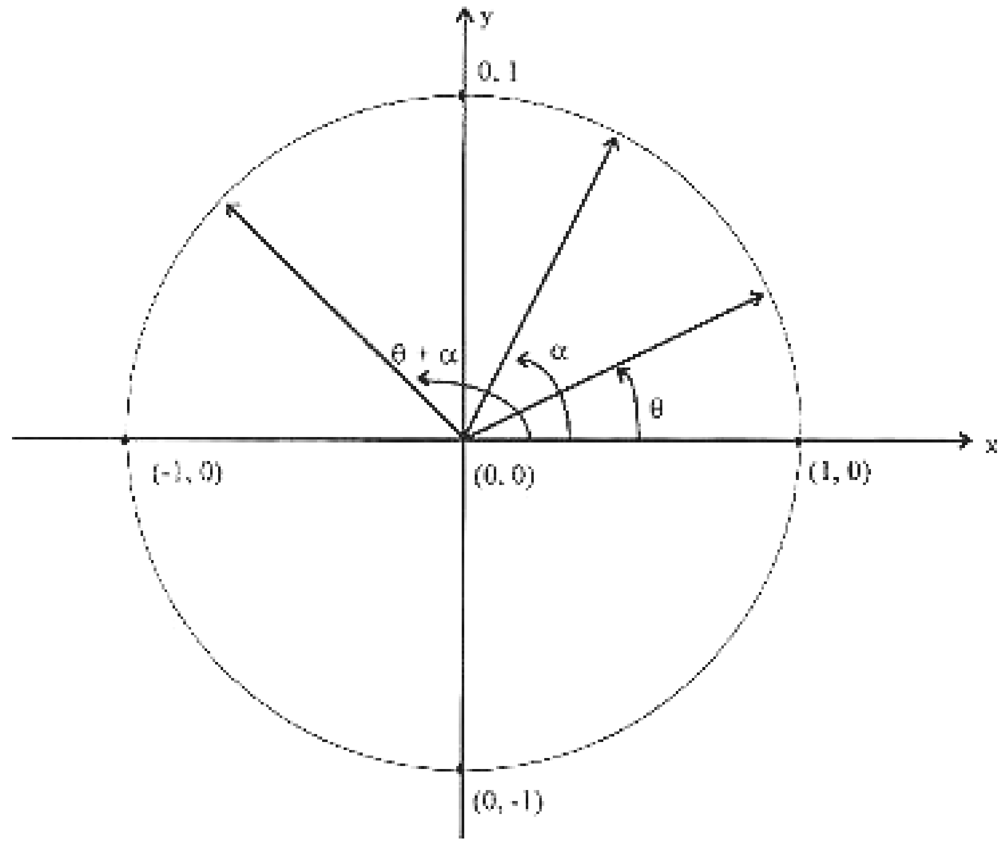


Figure 3.4. Vector multiplication.

From ‘An Imaginary Tale: The Story of ⌦⍰⍰’ by Paul J. Nahin
Princeton University Press, 1998

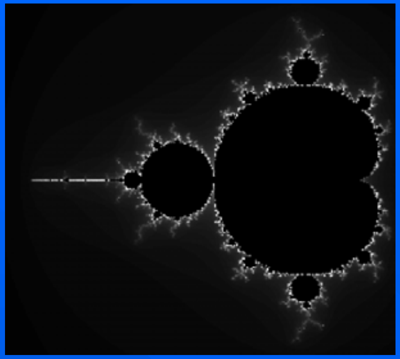
Preview: The Julia Set

- So when |z|=1, multiplying z by itself just takes you around and around on the unit circle.

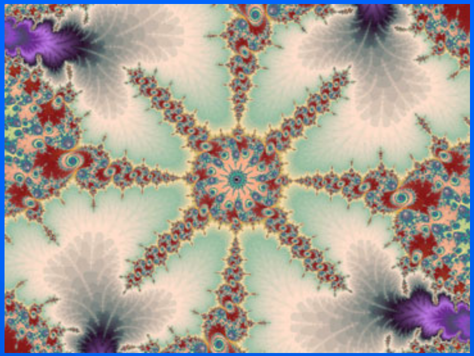
- Consider iterations on the complex plane using z values and a fixed c, in $z = z^2 + c$ (new z from old z).

- So if c = 0, points inside the unit circle converge to 0, and points outside diverge.

- The Julia Set is the boundary in between divergent and Convergent, i.e. is the unit circle itself when c = 0.



Mandelbrot Set



C = Complex Numbers

Seed Values: $z_0 = 0$, c in C

Function: $Q(z) = z^2 + c$

For all c, $|c| \odot 2$, compute {0,Q(0), Q(Q(0)), Q(Q(Q(0))),…} to some number of iterations N and determine whether the sequence is convergent, divergent or cyclic at that point.

The Mandelbrot Set consists of those points c in C for which the sequence does NOT diverge, when N goes to infinity.

Color code according to how quickly the value escapes (diverges)

Julia Set

Hold c constant for the entire picture, and make Z_0 be each point in the complex plane.

Julia Set is the boundary between divergent and not-divergent z Values (filled Julia includes not-divergent values)

If c is in the Mandelbrot Set, then the corresponding Julia Set is connected, otherwise it's Cantor Dust.

The Julia set looks similar to the Mandelbrot in the vicinity of c.