

Fractals Rule!

Some Background Information
Helpful to People Wanting to
Understand Fractal Geometry

Background Concepts

- Iterative Sequences
- Seed Values
- Convergence and Divergence
- Complex Numbers
- Complex Plane
- Multiplying Complex Numbers
- Mandelbrot Set and Julia Sets

Iterative Sequences

- Feed a number to a function and get the output.
- Apply the same function to output as new input.
- Continue...

$\{x, f(x), f(f(x)), f(f(f(x))), f(f(f(f(x))))\dots\}$

The *orbit* of x – Converges? Diverges? Cycles?

Seed Values: An Example

$$A_{n+1} = A_n + A_{n-1}$$

Seed Values: $A_0 = 0$ $A_1 = 1$

$$A_2 = 0 + 1 = 1$$

$$A_3 = 1 + 1 = 2$$

$$A_4 = 2 + 1 = 3$$

$$A_5 = 3 + 2 = 5$$

Sequence:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89...

Sequences in General

Classic examples:

Figurate Numbers

Triangular Numbers: 1, 3, 6, 10... $n(n+1)/2$

Square Numbers: 1, 4, 9, 25, 36... n^2

Cuboctahedral #s: 1, 12, 42, 92... $10n^2+2$

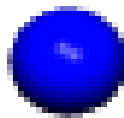
Primes $P(n)$: 2,3,5,7,11,13,17,19,23,29...

(sometimes no rule is known)

See:

Sloane's On-Line Dictionary of Integer Sequences

Cuboctahedral Numbers



Convergence / Divergence

Sequences defined by simple rules:

Additive (e.g. $A_{n+1} = A_n + A_{n-1}$)

Geometric (e.g. $z_n = r z_{n-1}$: 2, 4, 8, 16,...)

Other ($z_n = z_{n-1}^2 + c$)

Sequences defined by partial sums: running totals of elements

Divergent sequences: $\{1, 1+1/2, 1+1/2+1/3, 1+1/2+1/3+1/4, \dots\}$

Convergent sequences: $\{1, 1+1/4, 1+1/4+1/9, 1+1/4+1/9+1/16, \dots\}$

Cycling sequences: $\{\text{decimal digits of } 1/7\}$

Chaotic or aperiodic sequences: $\{\text{decimal digits of } \pi\}$

Side Topic: Converging Ratios

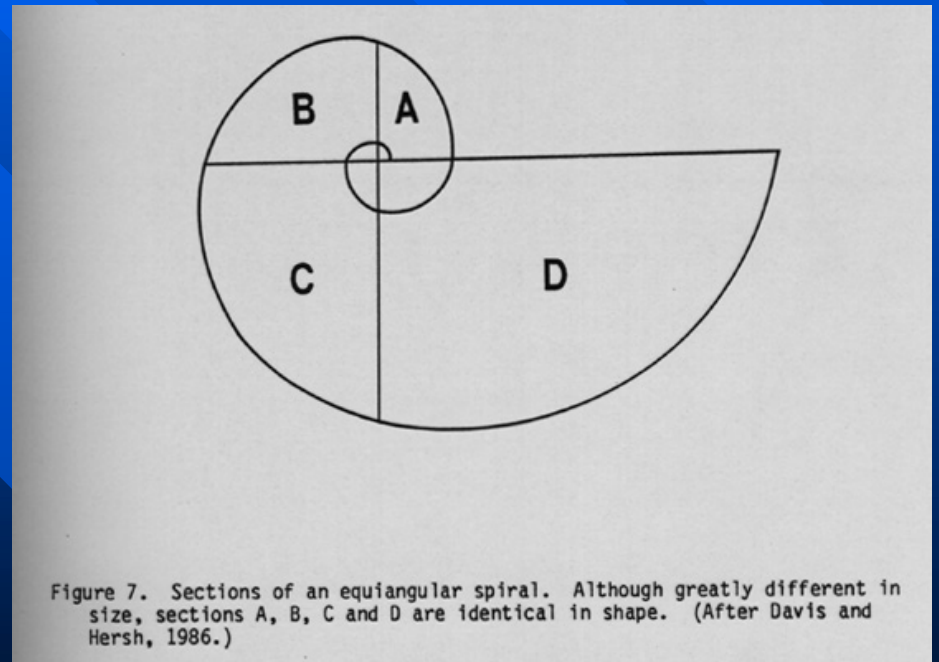
Sequence of Fibonacci #s:
0,1,1,2,3,5,8,13,21,34,55...

Ratios:
1/1, 2/1, 3/2, 5/3, 8/5, 13/8...

$\text{Fib}_{n+1}/\text{Fib}_n$ goes to ϕ
as n goes to infinity

$$\phi = (1 + \sqrt{5}) / 2$$

Fibonacci sequence is
both additive *and*
geometric



Self Similarity around a single point

Complex Numbers

Caspar Wessel (1745-1818)

Jean –Robert Argand (1768-1822)

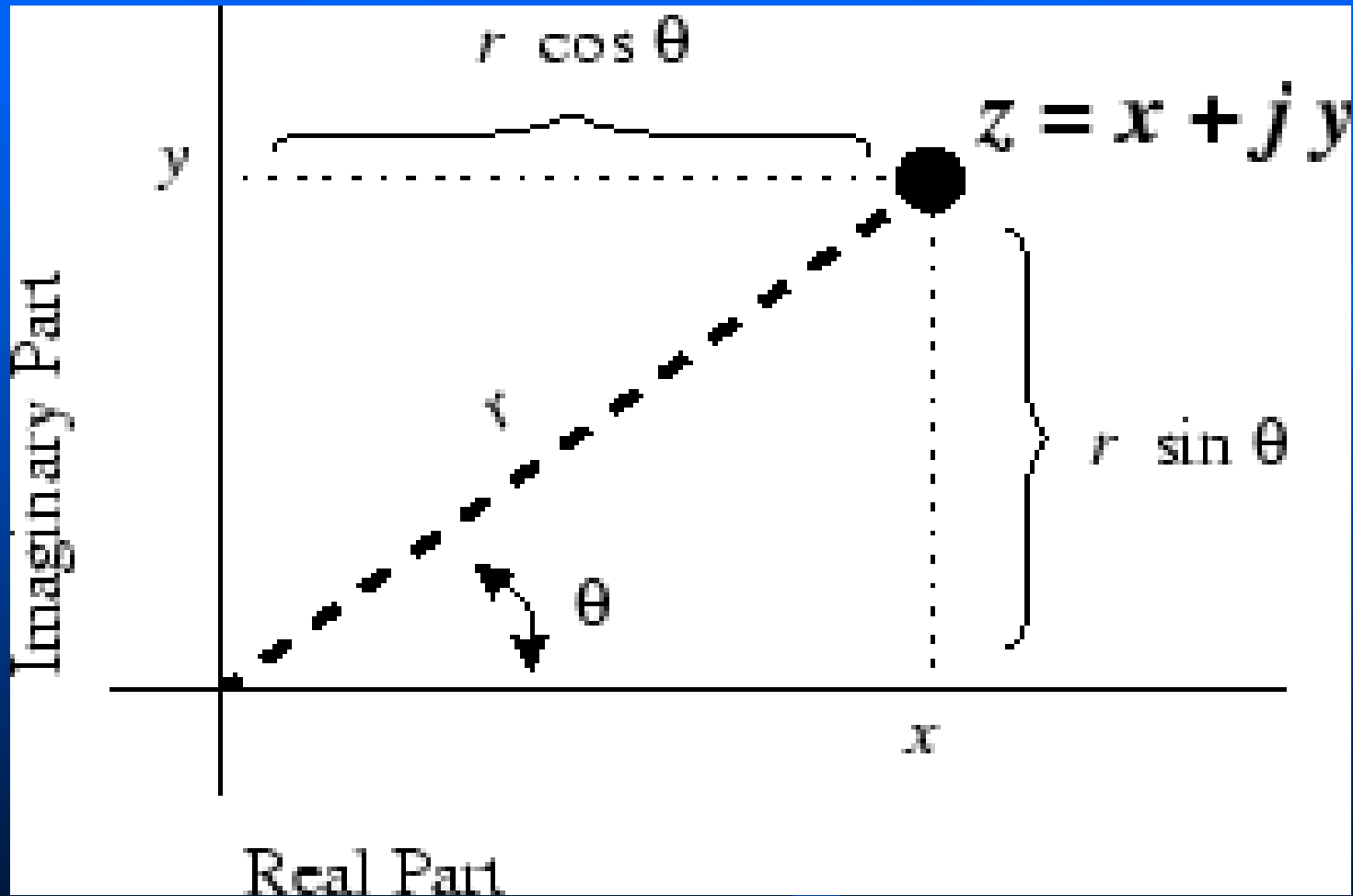
Abraham De Moivre (1667-1754)

Carl Friedrich Gauss (1777 – 1855)

Solutions to polynomials sometimes involved complex quantities even when all the solutions were real. Troubling to many.

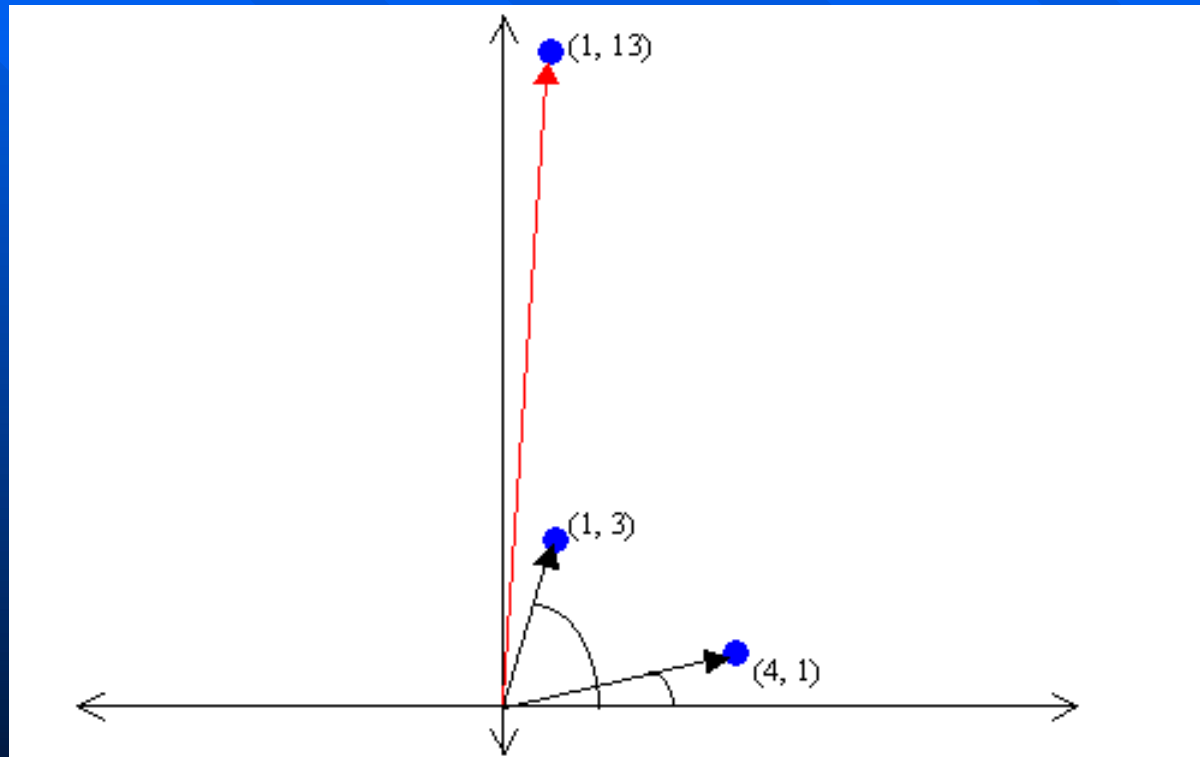
Generalization of polynomials required expansion of real numbers to this larger set allowing -1 to have even numbers of roots.

Argand Plane



$$Z_1 \times Z_2$$

Multiply Absolute Values, Add angles



More on Multiplying

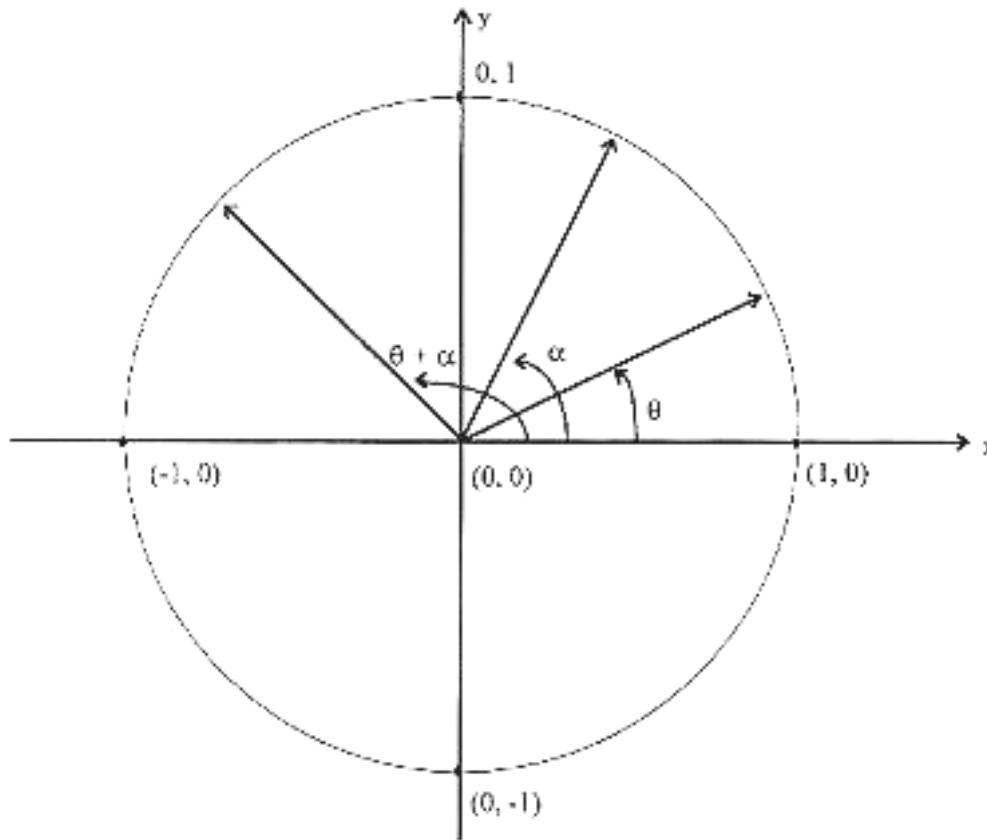
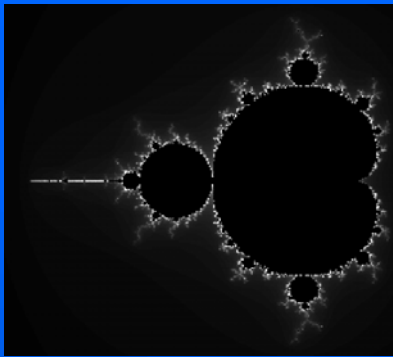


Figure 3.4. Vector multiplication.

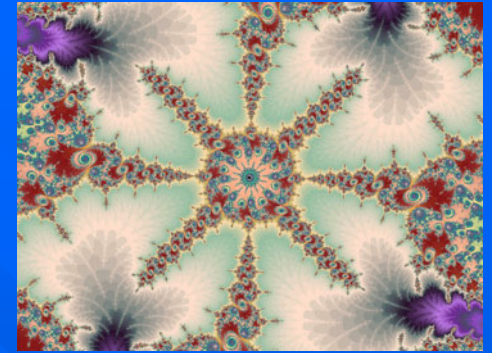
From 'An Imaginary Tale: The Story of \otimes \square \square ' by Paul J. Nahin
Princeton University Press, 1998

Preview: The Julia Set

- So when $|z|=1$, multiplying z by itself just takes you around and around on the unit circle.
- Consider iterations on the complex plane using z values and a fixed c , in $z = z^2 + c$ (new z from old z).
- So if $c = 0$, points inside the unit circle converge to 0, and points outside diverge.
- The Julia Set is the boundary in between divergent and Convergent, i.e. is the unit circle itself when $c = 0$.



Mandelbrot Set



\mathbb{C} = Complex Numbers

Seed Values: $z_0 = 0$, c in \mathbb{C}

Function: $Q(z) = z^2 + c$

For all c , $|c| \leq 2$, compute $\{0, Q(0), Q(Q(0)), Q(Q(Q(0))), \dots\}$ to some number of iterations N and determine whether the sequence is convergent, divergent or cyclic at that point.

The Mandelbrot Set consists of those points c in \mathbb{C} for which the sequence does NOT diverge, when N goes to infinity.

Color code according to how quickly the value escapes (diverges)

Julia Set

Hold c constant for the entire picture, and make Z_0 be each point in the complex plane.

Julia Set is the boundary between divergent and not-divergent z Values (filled Julia includes not-divergent values)

If c is in the Mandelbrot Set, then the corresponding Julia Set is connected, otherwise it's Cantor Dust.

The Julia set looks similar to the Mandelbrot in the vicinity of c .

